An efficient iteration for the extremal solutions of discrete-time algebraic Riccati equations

Chun-Yueh Chiang

Center for General Education,
National Formosa University, Huwei 632, Taiwan;
chiang@nfu.edu.tw
A joint work with Prof. H.Y. Fan (NTNU)

2021 TMS Annual Meeting
Jan.,17, 2022

Outline

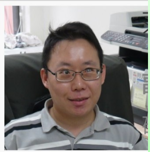
Reference

C.-Y. Chiang and H. Y. Fan* ,
An efficient iteration for the extremal solutions of discrete-time algebraic Riccati equations,
submitted for publication, (https://arxiv.org/abs/2111.08923), Nov., 2018.

C.-Y. Chiang*,
The convergence analysis of an accelerated iteration for solving algebraic Riccati equations,
Journal of the Franklin Institute, 359(1), pp:619–636, Jan.,2022.

M. M. Lin and C.-Y. Chiang*,
On the semigroup property for some structured iterations,
Journal of Computational and Applied Mathematics, ID:112768, Aug., 2020.

The series works with two Co-authors



Prof. Hung-Yuan, Fan, Department of Mathematics, NTNU, Taiwan.

Prof. Matthew M. Lin, Department of Mathematics, NCKU, Taiwan.

Outline

Beginning

In this talk we are mainly concerned with the extremal solutions of the discrete-time algebraic Riccati equation (DARE)

$$X = A^H XA - A^H XF_X + C^H C, \tag{1.1a}$$

or its equivalent expression

$$X = A^H X(I + GX)^{-1}A + H, \tag{1.1b}$$

where $F_X := (R + B^H XB)^{-1} B^H XA$, $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times m}$, $R \in \mathbb{C}^{m\times m}$ and $R > 0$, $C \in \mathbb{C}^{l\times n}$ with $m, l \leq n$, I is the identity matrix of compatible size, $G := BR^{-1}B^H \geq 0$ and $H = C^H C \geq 0$, respectively

The classification of solutions

Let the closed loop matrix $T_X := A - BF_X$ for any Hermitian solution X, the open unit disk by $\mathbb{D}$, the closed unit disk by $\bar{\mathbb{D}}$, the boundary of $\mathbb{D}$ by $\partial\mathbb{D}$, the region outside the open unit disk by $\mathbb{D}^c$.

1.With the shape of $\sigma(T_X)$:

1. Unmixed solution
   1. A subset Λ of $\mathbb{C}$ is Unmixed if $0 \in \Lambda$ and
   $$\Lambda \cap \widehat{\Lambda} = \partial\mathbb{D}, \Lambda \cup \widehat{\Lambda} = \mathbb{C},$$
   where $\widehat{\Lambda} := \{1/\bar{z}; z \in \Lambda \backslash \{0\}\}$.
   2. X is an unmixed solution if there exists an unmixed set Λ such that $\sigma(T_X) \subseteq \Lambda$.
2. Almost stabilizing solution: an unmixed solution with $\Lambda = \bar{\mathbb{D}}$.

The classification of solutions

$A > B$ (or $A \geq B$) if $A - B > 0$ (or $A - B \geq 0$).

2.With the Loewner order of the Hermitian solution set:

Four extremal solutions:

1. $X_{+,M}$: the maximal positive semidefinite solution.
2. $X_{+,m}$: the minimal positive semidefinite solution.
3. $X_{-,M}$: the maximal negative semidefinite solution.
4. $X_{-,m}$: the minimal negative semidefinite solution.

Motivation

Recent results

1. Newton method (P.Lancaster et al.): X_{NM}= maximal Hermitian solution.
2. Structure-preserving doubling algorithms (Lin W.W. et al.): X_{SDA}= (almost) stabilizing solution.
3. Maximal Hermitian solution= the (almost) stabilizing solution under the stabilizability condition.
4. $X_{+,M} = X_{+,m}$ under the stabilizability and detectability condition.(Chiang2021)
5. $X_{\mathrm{SDA}} = X_{+,m}$ even $\rho(T_{X_{\mathrm{SDA}}}) > 1$.(Chiang2021)

Motivation

Aims and Scope

1. Simple assumptions: (A, B) is stabilizable, $G \geq 0$ and $H \geq 0$.
2. Existence of four extremal solutions.
3. Based on the semigroup property, an accelerated fixed-point iteration (AFPI) is developed for solving the four extremal solutions of DARE.
4. AFPI works efficiently with R-superlinear convergence under the mild assumptions.
5. Comprehensive convergence analysis of AFPI.

Outline

Some basic and useful lemmas

Lemma:some fundamental identities

Let $X, \widehat{X} \in \mathrm{dom}(\mathcal{R}) := \{X \in \mathbb{H}_n \,|\, \det(R + B^H X B) \neq 0\}$ and the Stein operator $\mathcal{S}_A(X) := X - A^H X A$ for all $X \in \mathbb{H}_n$.

(i) If $A_F := A - BF$ for any $F \in \mathbb{C}^{m\times n}$ and $H_F := H + F^H R F$, then

$$X - \mathcal{R}(X) = \mathcal{S}_{A_F}(X) - H_F + K_F(X), \tag{2.1a}$$

where $K_F(X) := (F - F_X)^H (R + B^H X B)(F - F_X)$.

(ii) If $K(\widehat{X}, X) := K_{F_{\widehat{X}}}(X)$ and $H_{\widehat{X}} := H + F_{\widehat{X}}^H R F_{\widehat{X}}$, then (2.1a) can be rewritten as

$$X - \mathcal{R}(X) = \mathcal{S}_{T_{\widehat{X}}}(X) - H_{\widehat{X}} + K(\widehat{X}, X). \tag{2.1b}$$

Some basic and useful lemmas

Lemma: the spectral radius determining

Let $B \in \mathbb{C}^{n\times n}$ and $Q \geq 0$. If X_0 is a positive semidefinite solution of the Stein inequality $\mathcal{S}_B(X) \geq Q$, and $\mathrm{Ker}(Q) \subseteq \mathrm{Ker}(B-A)$ for some $A \in \mathbb{C}^{n\times n}$, then $\rho(B) \leq \max\{1, \rho(A)\}$. Furthermore, we have

(i) $\rho(B) \leq 1$ if $\rho(A) \leq 1$.

(ii) $\rho(B) < 1$ if $\rho(A) < 1$ or $\mathrm{Ker}(Q) \cap E_\lambda(B) = \{0\}$ for some $\lambda \in \sigma(B)$.

Lemma:definite constraint

Let $X \in \mathcal{R}_{\geq} := \{X \in \mathrm{dom}(\mathcal{R}) \,|\, X \geq \mathcal{R}(X)\}$. Then $R + B^H XB > 0$ and $K(\widehat{X}, X) \geq 0$ for any $\widehat{X} \in \mathbb{H}_n$.

The first kind of dual DARE

Assume that A is nonsingular. Let $X^{(A)} := A^{-H}XA^{-1}$ for any $X \in \mathbb{C}^{n\times n}$. It provides the formulation of the first kind of dual DARE

$$Y = \mathcal{D}_1(Y) := \widehat{H} + \widehat{A}^H Y(I + \widehat{G}Y)^{-1}\widehat{A},$$

where

$$\widehat{A} = A^{-1} - \widehat{B}\widehat{R}^{-1}B^H H^{(A)},$$
$$\widehat{G} = \widehat{B}\widehat{R}^{-1}\widehat{B}^H \geq 0,$$
$$\widehat{H} = H^{(A)} - B^H H^{(A)}\widehat{R}^{-1}H^{(A)}B,$$

where $\widehat{B} = A^{-1}B$ and $\widehat{R} = R + B^H H^{(A)}B$.

The first kind of dual DARE

Proposition

1. $Y = -X$ is a solution of $\mathcal{D}_1(Y)$ if and only if X is a solution of $\mathcal{R}(X)$. Furthermore,

$$(\mathcal{R}(X) - X)^{(A)} = (\mathcal{D}_1(Y) - Y)(I + GH^{(A)}).$$

2. $I - \widehat{G}X$ is nonsingular and

$$[(I + GX)^{-1}A] \times [(I + \widehat{G}Y)^{-1}\widehat{A}] = I$$
$$\sigma((I + \widehat{G}Y)^{-1}\widehat{A}) = \sigma(T_X^{-1}).$$

The second kind of dual DARE

Assume that A is nonsingular. Let nonsingular matrix $X \in \mathcal{R}_{=} := \{X \in \mathrm{dom}(\mathcal{R}) \,|\, X = \mathcal{R}(X)\}$. It can be shown that $Y := -X^{-1}$ satisfies

$$Y = \mathcal{D}_2(Y) := AY(I + HY)^{-1}A^H + G.$$

$$\left[X(I + GX)^{-1}A\right] \times \left[Y(I + HY)^{-1}A^H\right] = -I.$$

Proposition

1.
$$Y - \mathcal{D}_2(Y) = A\left[(X - H)^{-1} - (\mathcal{R}(X) - H)^{-1}\right] A^H,$$

2.
$$\sigma(((I + HY)^{-1}A^H)) = \sigma(XT_X^{-1}X^{-1}) = \sigma(T_X^{-1}).$$

Outline

Extremal solutions of the DARE

In this section, the existence of extremal solutions to the DARE (1.1) will be established iteratively through the Fixed-point iteration(FPIs) given by

$$X_{k+1} = \mathcal{R}(X_k),\ Y_{k+1} = \mathcal{D}_1(Y_k),\ Z_{k+1} = \mathcal{D}_2(Z_k),$$

with suitable X_0, Y_0 and Z_0.

FPI X_k for solving $X_{+,m}$ (Chiang2021)

Let $\rho_{\mathbb{D}}(M) := \max\{|\lambda| \mid \lambda \in \sigma(M) \cap \mathbb{D}\} < 1$.

1. Assumptions: $\mathcal{R}_{\geq} \cap \mathbb{N}_n \neq \emptyset$ and $\{X_k\}_{k=0}^{\infty}$ with $0 \leq X_0 \leq H$.
2. Result: $X_k \to X_{+,m}$ at least R-linearly. Furthermore,

$$\limsup_{k\to\infty} \sqrt[k]{\|X_k - X_{+,m}\|} \leq \rho_{\mathbb{D}}(T_{X_{+,m}})^2 < 1.$$

Extremal solutions of the DARE

FPI X_k for solving $X_{+,M}$

1. Assumptions: $\rho(T_{X_\star}) < 1$ for some $X_\star \in \mathbb{H}_n$ and $\{X_k\}_{k=0}^{\infty}$ with $X_0 = \mathcal{S}_{T_{X_\star}}^{-1}(H_{X_\star}) \in \mathcal{S}_{\geq}$.
2. Result:
   a. $\mathcal{S}_{\geq} := \{X \in \mathbb{H}_n \,|\, \mathcal{S}_{T_{X_\star}}(X) \geq H_{X_\star}\} \subseteq \mathcal{R}_{\geq} \cap \mathbb{N}_n$.
   b. $X_k \to X_{+,M}$ at least R-linearly if $\rho(T_{+,M}) < 1$. Furthermore,
   $$\limsup_{k\to\infty} \sqrt[k]{\|X_k - X_{+,M}\|} \leq \rho(T_{+,M})^2.$$
   c. For each $k \geq 0$, $\rho(T_{X_k}) < 1$ and thus $\rho(T_{+,M}) \leq 1$.

Extremal solutions of the DARE

Some equivalent conditions for the stabilizability of the pair (A, B)

The following statements are equivalent:

(i) The pair (A, B) is stabilizable.

(ii) Tere exists a matrix $X_\star \in \mathbb{H}_n$ satisfying $\rho(T_{X_\star}) < 1$.

(iii) The DARE (1.1) has a unique almost stabilizing solution $X \in \mathbb{H}_n$.

(iv) The DARE (1.1) has a maximal and almost stabilizing solution $X \in \mathbb{H}_n$.

Extremal solutions of the DARE

FPI Y_k for solving $X_{-,M}, X_{-,m}$

Assume that $\widehat{H} \geq 0$, A is nonsingular, $\mu(M) := \min\{|\lambda| \mid \lambda \in \sigma(M)\}$.

1. Assumptions: $\mathcal{D}_{\geq}^{(1)} \cap \mathbb{N}_n \neq \emptyset$ and $\{Y_k\}_{k=0}^{\infty}$ with $0 \leq Y_0 \leq \widehat{H}$.
2. Result: $Y_k \to -X_{-,M}$ at least R-linearly. Furthermore,

$$\limsup_{k\to\infty} \sqrt[k]{\|Y_k + X_{-,M}\|} \leq \rho_{\mathbb{D}}(T_{X_{-,M}}^{-1})^2 < 1.$$

1. Assumptions: $\mathrm{rank}[A - \lambda I \ \ B] = n$ for all $\lambda \in \bar{\mathbb{D}}\backslash\{0\}$ and $\{Y_k\}_{k=0}^{\infty}$ with $Y_0 = \mathcal{S}_{\widehat{A}_{\widehat{F}}}^{-1}(\widehat{H}_{\widehat{F}})$.
2. Result: $Y_k \to -X_{-,m}$,

$$\limsup_{k\to\infty} \sqrt[k]{\|Y_k + X_{-,m}\|} \leq \rho(T_{X_{-,m}}^{-1})^2 = \mu(T_{X_{-,m}})^2 \leq 1.$$

Extremal solutions of the DARE

FPI Z_k for solving $X_{-,m}$

1. Assumptions: $D^{(2)}_{\geq} \cap \mathbb{N}_n \neq \emptyset$, (A, B) is controllable and $\{Z_k\}_{k=0}^{\infty}$ with $Z_0 = 0$.
2. Result: $Z_k \to -X_{-,m}$ at least R-linearly. Furthermore,

$$\limsup_{k\to\infty} \sqrt[k]{\|Z_k - Z_\infty\|} \leq \rho_{\mathbb{D}}(T_{Z_\infty}^{-1})^2 < 1.$$

Furthermore, the minimal negative semidefinite solution of DARE (1.1) can be obtained by $X_{-,m} = -Z_\infty^{-1}$.

Outline

Acceleration of fixed-point iteration

Idea

1. In this section, for any positive integer $r > 1$, we will revisit an accelerated FPI (AFPI) of the form

$$\widehat{X}_{k+1} = \mathcal{R}^{(r^{k+1}-r^k)}(\widehat{X}_k), \quad k \geq 1,$$
$$\widehat{X}_1 = \mathcal{R}^{(r)}(\widehat{X}_0), \quad k = 1$$

   with $\widehat{X}_0 = X_0$.

2. Theoretically, the iteration of the above form is equivalent to the formula

$$\widehat{X}_k = \mathcal{R}^{(r^k)}(\widehat{X}_0) = X_{r^k}, \quad k \geq 1,$$

   with $\widehat{X}_0 = X_0$.

Equivalent formulation of the fixed-point iteration

The following definition modifies the semigroup property of the iteration associated a binary operator.

Definition:

Let $\mathbb{K}_n \subseteq \mathbb{C}^{p\times q}$ and $F : \mathbb{K}_n \times \mathbb{K}_n \to \mathbb{K}_n$ be a binary matrix operator, where p and q are positive integers. We call that an iteration

$$\mathbb{X}_{k+1} = F(\mathbb{X}_k, \mathbb{X}_0), \quad k \geq 0,$$

has the semigroup property if the operator F satisfies the following associative rule:

$$F(F(Y, Z), W) = F(Y, F(Z, W))$$

for any Y, Z and W in $\mathbb{K}_n$.

The semigroup property of $X_{k+1} = F(X_k, X_0)$

Example: A binary operator F with semigroup property

Let A be an arbitrary matrix with size $n \times n$. For any three n-square matrices X, Y and Z, we assume that $A + X + Y$ and $A + Y + Z$ are nonsingular. Let $\Delta_{X,Y} = (A + X + Y)^{-1}$ and the binary matrix function F be defined by $F(X, Y) = X\Delta_{X,Y}Y$.

Theorem :The discrete flow property

Given an iteration $X_{k+1} = F(X_k, X_0)$ with semigroup property, then

$$X_{i+j+1} = F(X_i, X_j)$$

for any nonnegative integers i and j.

AFPI for solving DARE

1. The fixed-point iteration X_k can be rewritten as the following formulation

$$X_{k+1} = \mathcal{R}^{(k)}(\mathcal{R}(X_0)) = \mathcal{R}^{(k+1)}(X_0) = H_k + A_k^H X_0 (I + G_k X_0)^{-1} A_k,$$

where the sequence of matrices $\{(A_k, G_k, H_k)\}_{k=0}^{\infty}$ is generated by $\mathbb{X}_{k+1} = F(\mathbb{X}_k, \mathbb{X}_0)$ with $\mathbb{X}_k := \begin{bmatrix} A_k^H & G_k & H_k \end{bmatrix}^H$ and $\mathbb{X}_0 := \begin{bmatrix} A^\top & G^\top & H^\top \end{bmatrix}^\top$ for each $k \geq 0$.

2. $F : \mathbb{K}_n \times \mathbb{K}_n \to F(\mathbb{X}_k, \mathbb{K}_n$ is a binary operator defined by

$$F(U, V) := \begin{bmatrix} V_1 \Delta_{U_2, V_3} U_1 \\ V_2 + V_1 \Delta_{U_2, G_3} U_2 V_1^H \\ U_3 + U_1^H V_3 \Delta_{U_2, V_3} U_1 \end{bmatrix}, \qquad (4.1)$$

with $U, V \in \mathbb{K}_n := \mathbb{C}^{n \times n} \times \mathbb{H}_n \times \mathbb{H}_n$, $\Delta_{U_2, V_3} = (I + U_2 V_3)^{-1}$.

The accelerated fixed-point iteration

1. The operator $\mathbf{F}_\ell : \mathbb{K}_n \to \mathbb{K}_n$ is defined recursively by

$$\mathbf{F}_{\ell+1}(\mathbb{X}) = F(\mathbb{X}, \mathbf{F}_\ell(\mathbb{X})), \quad \ell \geq 1,$$

with $\mathbf{F}_1(\mathbb{X}) = \mathbb{X}$ for all $\mathbb{X} \in \mathbb{K}_n$

2. $$\mathbf{X}_{k+1} = \mathbf{F}_r(\mathbf{X}_k), \quad k \geq 0,$$

with $\mathbf{X}_0 := \begin{bmatrix} A^H & G & H \end{bmatrix}^H$, for constructing $\mathbf{A}_k = A_{r^k-1}$, $\mathbf{G}_k = G_{r^k-1}$ and $\mathbf{H}_k = H_{r^k-1}$, respectively.

The Accelerated Fixed-Point Iteration with r (AFPI(r))

1. Given a positive integer $r > 1$,let $\widehat{X}_0 = X_0$;
2. Outer iteration: For $k = 1, \ldots,$ iterate

$$\mathbf{X}_{k+1} = F(\mathbf{X}_k, \mathbf{X}_k^{(r-1)}) = \begin{bmatrix} \mathbf{A}_{k+1} & \mathbf{G}_{k+1} & \mathbf{H}_{k+1} \end{bmatrix}^\top,$$
$$\widehat{X}_{k+1} = \mathbf{A}_{k+1}^H \widehat{X}_0 (I + \mathbf{G}_{k+1}\widehat{X}_0)^{-1} \mathbf{A}_{k+1} + \mathbf{H}_{k+1}$$

until convergence, where $\mathbf{X}_k^{(r-1)}$ is defined in step 3.

3. Inner iteration: For $\ell = 1, \ldots, r-2$, iterate

$$\mathbf{X}_k^{(\ell+1)} = F(\mathbf{X}_k, \mathbf{X}_k^{(\ell)}),$$

with $\mathbf{X}_k^{(1)} = \mathbf{X}_k$.

Convergence analysis of the AFPI

Applying AFPI to X_k for solving $X_{+,m}$ and X_{+M}

Based on FPI: X_k and the same hypotheses:

(i) $\{\mathbf{H}_k\}_{k=0}^{\infty}$ converges at least R-superlinearly to $X_{+,m}$ with the rate of convergence

$$\limsup_{k\to\infty} \sqrt[r^k]{\|\mathbf{H}_k - X_{+,m}\|} \leq \rho_{\mathbb{D}}(T_{X_{+,m}})^2 < 1.$$

(ii) $\{\widehat{X}_k\}_{k=0}^{\infty}$ converges to $X_{+,M}$ with the rate of convergence

$$\limsup_{k\to\infty} \sqrt[r^k]{\|\widehat{X}_k - X_{+,M}\|} \leq \rho(T_{X_{+,M}})^2,$$

Convergence analysis of the AFPI

Applying AFPI to Y_k for solving $X_{-,m}$ and X_{-M}

Based on FPI: Y_k and the same hypotheses, applying AFPI to first kind of DARE:

(i) $\{\mathbf{H}_k\}_{k=0}^{\infty}$ converges at least R-superlinearly to $-X_{-,M}$ with the rate of convergence

$$\limsup_{k\to\infty} \sqrt[r^k]{\|\mathbf{H}_k + X_{-,M}\|} \leq \rho_{\mathbb{D}}(T_{X_{-,M}}^{-1})^2 < 1.$$

(ii) $\{\widehat{X}_k\}_{k=0}^{\infty}$ converges at least R-superlinearly to $-X_{-,m}$ with the rate of convergence

$$\limsup_{k\to\infty} \sqrt[r^k]{\|\widehat{X}_k + X_{-,m}\|} \leq \rho(T_{X_{-,m}}^{-1})^2 = \mu(T_{X_{-,m}})^{-2}.$$

Convergence analysis of the AFPI

Applying AFPI to Z_k for solving $X_{-,m}$

Based on FPI: Z_k and the same hypotheses:

- We have $\mathbf{G}_k = Z_{r^k-1}$. Furthermore, $\{\mathbf{G}_k\}_{k=0}^{\infty}$ converges at least R-superlinearly to the unique almost stabilizing solution $G_\infty = -X_{-,m} > 0$ of the second kind of dual DARE with the rate of convergence

$$\limsup_{k\to\infty} \sqrt[r^k]{\|\mathbf{G}_k + X_{-,m}\|} \leq \rho_{\mathbb{D}}(T_{G_\infty}^{-1})^2 < 1.$$

Numerical examples

Environment setting

In this section, we present four examples to illustrate the accuracy and efficiency of the AFPI(r) for solving the extremal solutions of the DARE.

1. In the first three examples we compared the AFPI algorithm, through the sequence $\{\widehat{X}_k\}_{k=0}^{\infty}$ starting with some suitable initial $\widehat{X}_0$, with Newton's method (NTM) for solving the maximal or (almost) stabilizing solution $X_{+,M} \geq 0$ of DARE (1.1b).
2. $\widehat{X}_0 \geq 0$ is the unique solution of Stein matrix equation $\mathcal{S}_{A_F}(X) = H + F^H RF$, which can be computed by MATLAB command `dlyap` directly.
3. $R = I$ in all numerical examples.

Numerical examples

Environment setting

1. For an approximate solution Z to the DARE (1.1), we will report its normalized residual

$$NRes(Z) := \frac{\|Z - \mathcal{R}(Z)\|}{\|Z\| + \|A^H Z(I + GZ)^{-1} A\| + \|H\|},$$

2. $$T_Z := (I + GZ)^{-1} A, \quad \mu(T_Z) := \min\{|\lambda| \mid \lambda \in \sigma(T_Z)\}.$$

3. We terminated the numerical methods AFPI and NTM when $NRes \leq 1.0 \times 10^{-15}$ in Example1–3, and the AFPI algorithm terminated when $NRes \leq 1.0 \times 10^{-12}$ in Example 4, respectively.

EX1

1. Let the coefficient matrices of DARE (1.1b) be given by

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 1/2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad H = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

   Then it is easily seen that the pair (A, B) is stabilizable, but (A, C) is not detectable.

2. Only two positive semidefinite solutions, namely,

$$X_{+,M} = \begin{bmatrix} 8 & 0 \\ 0 & 4/3 \end{bmatrix}, \quad X_{+,m} = \begin{bmatrix} 0 & 0 \\ 0 & 4/3 \end{bmatrix}.$$

3. The matrix $X_{+,M}$ is the maximal and stabilizing solution of the DARE such that the eigenvalues of $T_{X_{+,M}} = (I + GX_{+,M})^{-1}A$ are $1/3$ and $1/2$, and $\sigma(T_{X_{+,m}}) = \sigma(A)$, respectively. Thus, $X_{+,m}$ is the minimal positive semidefinite solution of the DARE (1.1b) with the property $\rho(T_{X_{+,m}}) = 3 > 1$.

EX1

1

k	$NRes(\widehat{X}_k)$	$NRes(\mathbf{H}_k)$	$\rho(T_{\widehat{X}_k})$	$\rho(T_{\mathbf{H}_k})$	$\|\mathbf{A}_k\|$
1	5.7×10^{-4}	2.4×10^{-2}	5.0×10^{-1}	3.0×10^{0}	9.0×10^{0}
2	7.1×10^{-6}	1.5×10^{-3}	5.0×10^{-1}	3.0×10^{0}	8.1×10^{1}
3	1.1×10^{-9}	5.7×10^{-6}	5.0×10^{-1}	3.0×10^{0}	6.6×10^{3}
4	1.0×10^{-16}	8.7×10^{-11}	5.0×10^{-1}	3.0×10^{0}	4.3×10^{7}
5		0.0×10^{0}		3.0×10^{0}	1.9×10^{15}

Table: Numerical results of AFPI(2) for EX1.

2

k	$NRes(X_k)$	$\rho(T_{X_k})$
1	5.7×10^{-4}	5.0×10^{-1}
2	8.7×10^{-8}	5.0×10^{-1}
3	2.1×10^{-15}	5.0×10^{-1}
4	0.0×10^{0}	5.0×10^{-1}

Table: Numerical results of NTM for EX1.

Ex2

In this example we consider the DARE (1.1b) with its 5×5 coefficient matrices being defined by

$$A = \begin{bmatrix} 2.9 & 1 \\ 0 & 2.9 \end{bmatrix} \oplus 0_2 \oplus 1, \quad H = 0_2 \oplus \begin{bmatrix} 200 & -0.5 \\ -0.5 & 200 \end{bmatrix} \oplus 1$$

and $B = \mathrm{diag}(\sqrt{2}, 1, 0, 0, 1)$, respectively.

1. It can be shown that the explicit solution $X_{+,m}$ of the DARE (1.1b) is
$$X_{+,m} = 0_2 \oplus \begin{bmatrix} 200 & -0.5 \\ -0.5 & 200 \end{bmatrix} \oplus (1+\sqrt{5})/2,$$
which is almost the same as H except the $(5, 5)$-entry.
2. (A, B) is stabilizable and thus $X_{+,M}$ exists.

Ex2

1

k	$NRes(\widehat{X}_k)$	$NRes(\mathbf{H}_k)$	$\rho(T_{\widehat{X}_k})$	$\rho(T_{\mathbf{H}_k})$	$\|\mathbf{A}_k\|$
1	9.0×10^{-5}	2.5×10^{-4}	3.8×10^{-1}	2.9×10^{0}	1.2×10^{1}
2	1.7×10^{-6}	5.6×10^{-6}	3.8×10^{-1}	2.9×10^{0}	1.3×10^{2}
3	7.1×10^{-10}	2.6×10^{-9}	3.8×10^{-1}	2.9×10^{0}	1.5×10^{4}
4	7.2×10^{-17}	5.2×10^{-16}	3.8×10^{-1}	2.9×10^{0}	1.4×10^{8}

Table: Numerical results of AFPI(2) for EX2.

2

k	$NRes(X_k)$	$\rho(T_{X_k})$
1	3.4×10^{-4}	3.8×10^{-1}
2	1.3×10^{-6}	3.8×10^{-1}
3	2.6×10^{-11}	3.8×10^{-1}
4	2.3×10^{-18}	3.8×10^{-1}

Table: Numerical results of NTM for EX2.

EX3

This example is modified from Example 6.2 of [GuoSIMAX98]. For $\varepsilon \geq 0$, the coefficient matrices of DARE (1.1b) are defined by

$$A = \mathrm{diag}\left(\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & \varepsilon \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{-1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & 1 & 0 \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \right),$$

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & \varepsilon & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \quad H = C^H C = 0 \in \mathbb{R}^{8\times 8}.$$

Ex3

1. This DARE has a unique positive semidefinite solution $X_{+,M} = X_{+,m} = 0$, where $X_{+,M}$ is the almost stabilizing solution with $\sigma(T_{X_{+,M}}) = \sigma(A)$ for all $\varepsilon \geq 0$.
2. Note that this DARE is just the same as the one appeared in Example 6.2 of [GUOSIMAX98] when $\varepsilon = 0$, in which all unimodular eigenvalues of A are semisimple.
3.

Method	Iter. No.	CPU Time (sec.)
AFPI(2)	50	8.78×10^{-3}
AFPI(4)	25	1.45×10^{-2}
AFPI(8)	17	1.35×10^{-2}
AFPI(100)	8	1.51×10^{-2}
NTM	50	3.08×10^{-2}

Table: The CPU times of numerical methods for EX3 with $\varepsilon = 0$.

EX3

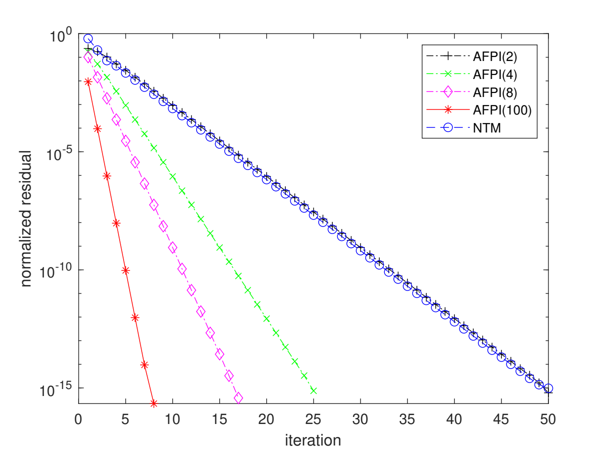


Figure: Convergence histories of numerical methods for EX3 with $\varepsilon = 0$.

EX3

k	$\|\widehat{X}_k - X_{+,M}\|$	$\frac{\|\widehat{X}_k - X_{+,M}\|}{\|\widehat{X}_{k-1} - X_{+,M}\|}$	$\rho(T_{\widehat{X}_k})$	$\|\mathbf{A}_k\|$
1	2.0×10^{-2}	1.06×10^{-3}	9.90×10^{-1}	1.0×10^{2}
2	2.0×10^{-4}	1.02×10^{-2}	1.0×10^{0}	1.0×10^{4}
3	2.0×10^{-6}	1.00×10^{-2}	1.0×10^{0}	1.0×10^{6}
4	2.0×10^{-8}	1.00×10^{-2}	1.0×10^{0}	1.0×10^{8}
5	2.0×10^{-10}	1.00×10^{-2}	1.0×10^{0}	1.0×10^{10}
6	2.0×10^{-12}	1.00×10^{-2}	1.0×10^{0}	1.0×10^{12}
7	2.0×10^{-14}	1.00×10^{-2}	1.0×10^{0}	1.0×10^{14}

Table: Numerical results of AFPI(100) for EX3 with $\varepsilon = 1$.

EX4

1. This example will demonstrate the feasibility of our AFPI algorithm for solving the negative semidefinite extremal solutions. As quoted from Example 6.2 of [IJC2017],

$$A = \begin{bmatrix} 4 & 3 \\ \frac{-9}{2} & \frac{-7}{2} \end{bmatrix}, \quad B = \begin{bmatrix} 6 \\ -5 \end{bmatrix}, \quad H = \begin{bmatrix} 9 & 6 \\ 6 & 4 \end{bmatrix}.$$

2. $$\widehat{A} = \begin{bmatrix} 7 & 6 \\ -9 & -8 \end{bmatrix}, \quad \widehat{B} = \begin{bmatrix} 12 \\ -14 \end{bmatrix}, \quad \widehat{C} = \begin{bmatrix} 24 & 16 \end{bmatrix}, \quad \widehat{R} = 65, \quad \widehat{H} = H.$$

EX4

1. This DARE has three extremal solutions, namely,

$$X_{+,M} = X_{+,m} = \begin{bmatrix} \frac{9}{2} + \frac{9}{8}\sqrt{17} & 3 + \frac{3}{4}\sqrt{17} \\ 3 + \frac{3}{4}\sqrt{17} & 2 + \frac{\sqrt{17}}{2} \end{bmatrix}$$

and

$$X_{-,M} = \begin{bmatrix} \frac{9}{2} - \frac{9}{8}\sqrt{17} & 3 - \frac{3}{4}\sqrt{17} \\ 3 - \frac{3}{4}\sqrt{17} & 2 - \frac{\sqrt{17}}{2} \end{bmatrix}, \quad X_{-,m} = \begin{bmatrix} \frac{-103}{12} - \frac{\sqrt{17}}{8} & \frac{-39}{4} - \frac{\sqrt{17}}{4} \\ \frac{-39}{4} - \frac{\sqrt{17}}{4} & \frac{-43}{4} - \frac{\sqrt{17}}{2} \end{bmatrix}.$$

2. $X_{+,M} \geq 0$ is the maximal and stabilizing solution, $X_{-,M}$ is the maximal negative semidefinite solution and $X_{-,m} \leq 0$ is the minimal solution, respectively.

EX4

k	$NRes(-\mathbf{H}_k)$	$NRes(-\widehat{X}_k)$	$\mu(T_{-\mathbf{H}_k})$	$\mu(T_{-\widehat{X}_k})$	$\|\mathbf{A}_k\|$
1	2.1×10^{-13}	9.5×10^{-1}	5.0×10^{-1}	2.0×10^{0}	5.7×10^{1}
2	2.1×10^{-13}	5.1×10^{-8}	5.0×10^{-1}	2.0×10^{0}	2.3×10^{5}
3	2.1×10^{-13}	7.4×10^{-13}	5.0×10^{-1}	2.0×10^{0}	1.3×10^{-1}

Table: Numerical results of AFPI(4) for EX4.

Outline

Concluding Remark

1. In most of the past works, it is always assumed that the DARE has a unique maximal or (almost) stabilizing solution X with $\rho(T_X) \leq 1$ and another meaningful solutions are lacking in brief discussion. Our contribution fills in the existing gap in finding four extremal solutions of the DARE.
2. Theoretically, we provides an accelerated technique, embedded with a discrete-type flow property, to solve the four extremal solutions. This property then allows us to advance the original fixed-point iterative method.
3. Generally speaking, the convergence speed of accelerated iteration has R-order r, and even more, for the singular case, the iteration still succeeds with a linear rate of convergence.
4. How to apply the accelerated techniques in the work for solving unmixed solution leads to the work in future.

Thank you for your attention!